Ex 1. 1 *(Coin flips) A fair coin is flipped until the first head occurs. Let X denote the number of flips required.*

- *Find the entropy $H(X)$ in bits.*

Ex 1. 2 *Let X be a random variable taking on a finite number of values. What is the (general) inequality relationship of $H(X)$ and $H(Y)$ if (a) $Y = 2X$ (b) $Y = \cos X$:*

Ex 1. 3 *What is the minimum value of $H(p_1, \ldots, p_n) = H(\mathbf{p})$ as $\mathbf{p}$ ranges over the set of n-dimensional probability vectors? Find all $\mathbf{p}$'s which achieve this minimum.*

Ex 1. 6 *Let X and Y be random variables that take on values $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_s$, respectively. Let $Z = X + Y$.*

- *a= Show that $H(Z|X) = H(Y|X)$. Argue that if X, Y are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus the addition of independent random variables adds uncertainty.*
- *b) Give an example of (necessarily dependent) random variables in which $H(Y) \geq H(Z)$ and $H(X) \geq H(Z)$.*

- *c) Under what conditions does $H(Z) = H(X) + H(Y)$?*

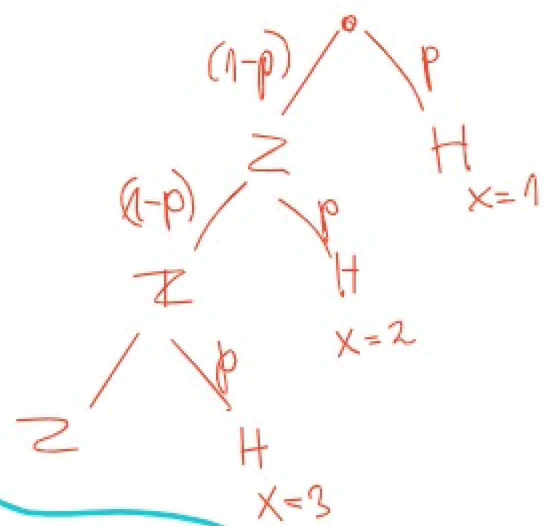
X = number of flips until first head

$\mathbb{P}(\text{head}) = p \in (0,1)$

$\mathbb{P}(X=1) = p$

$\mathbb{P}(X=2) = (1-p) \cdot p$

$\mathbb{P}(X=3) = (1-p)^2 \cdot p$



1st flip

2nd flip

$$\mathbb{P}(X=k) = (1-p)^{k-1} \cdot p$$

$k \in \mathbb{N} = \{1,2,3,\ldots\}$

$X \sim \text{Geo}(p)$

Geometric distribution

$\log(a \cdot b) = \ell$

Geometric distribution

$$H(X) = -\sum_{k\in X} p_X(k)\cdot \log(p_X(k))$$

$$= -\sum_{k\in\mathbb{N}_{\geq 1}} (1-p)^{k-1}\cdot p\cdot \log\left((1-p)^{k-1}\cdot p\right)$$

$$= -p\cdot\sum_{k\in\mathbb{N}_{\geq 1}}(1-p)^{k-1}\cdot\left(\log(p)+(k-1)\cdot\log(1-p)\right)$$

$$= -\frac{1}{2}\cdot\sum_{k\in\mathbb{N}}\left(\frac{1}{2}\right)^{k-1}\cdot\left(\underbrace{\log\left(\frac{1}{2}\right)}_{=-1}+(k-1)\cdot\underbrace{\log\left(\frac{1}{2}\right)}_{=-1}\right)$$

$$= \sum_{k\in\mathbb{N}}\left(\frac{1}{2}\right)^{k-1}\cdot\left(-1+(k-1)\cdot(-1)\right)$$

$$= -\sum_{k\in\mathbb{N}}\left(\frac{1}{2}\right)^{k}\cdot(-k)$$

$$= \sum_{k\in\mathbb{N}} k\cdot(p)^k = \sum_{k\in\mathbb{N}}\underbrace{\left(\frac{\partial}{\partial p}p^k\right)}_{k\cdot p^{k-1}}\cdot p$$

$$= p\cdot\frac{\partial}{\partial p}\left(\sum_{k\in\mathbb{N}_{\geq 1}} p^k\right)$$

$$= p\cdot\frac{\partial}{\partial p}\left(\sum_{k\in\mathbb{N}_0}p^k - \overset{p^0}{\overset{\shortparallel}{1}}\right)$$

$$= p\cdot\frac{\partial}{\partial p}\cdot\left(\frac{1}{1-p}-1\right)$$

$$= \frac{p}{(1-p)^2} =$$

$$= \frac{\frac{1}{2}}{(1-\frac{1}{2})^2}$$

$$= \underline{\underline{2}}$$

$\log(a\cdot b) = \log(a)+\log(b)$

$\log(a^b) = b\cdot\log a$

$\sum_{i\in J} c\cdot a_i = c\cdot\sum_{i\in J}a_i$

$\log\left(\frac{1}{2}\right) = -1$ $\quad 2^{\boxed{-1}} = \frac{1}{2}$

$$\boxed{\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}}$$

Geometric series

$(x^{10})' = 10\cdot x^9$

$(x^{100})' = 100\cdot x^{99}$

$(x^n)' = n\cdot x^{n-1}$

$\frac{\partial}{\partial p}(p^k) = k\cdot p^{k-1}$

$\left(\frac{f}{g}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2}$

$\left(\frac{1}{1-x}\right)' = \frac{0\cdot(1-x)-1\cdot(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$

$(a\cdot f)' = a\cdot f'$

$(-x)' = -(x)' = -(x^1)' =$
$= -1\cdot \underset{1}{x^0} = -1$

$$\sum_{n=0} n r^n = \frac{r}{(1-r)^2}$$

Ex 1. 2 *Let X be a random variable taking on a finite number of values. What is the (general) inequality relationship of $H(X)$ and $H(Y)$ if (a) $Y = 2X$ (b) $Y = \cos X$?*

Ex 1. 3 *What is the minimum value of $H(p_1, \dots, p_n) = H(\mathbf{p})$ as $\mathbf{p}$ ranges over the set of n-dimensional probability vectors? Find all $\mathbf{p}$'s which achieve this minimum.*

1.3. $X \in \{x_1, \dots, x_n\}$

$\mathbb{P}(X = x_1) = p_1, \dots, \mathbb{P}(X = x_n) = p_n$

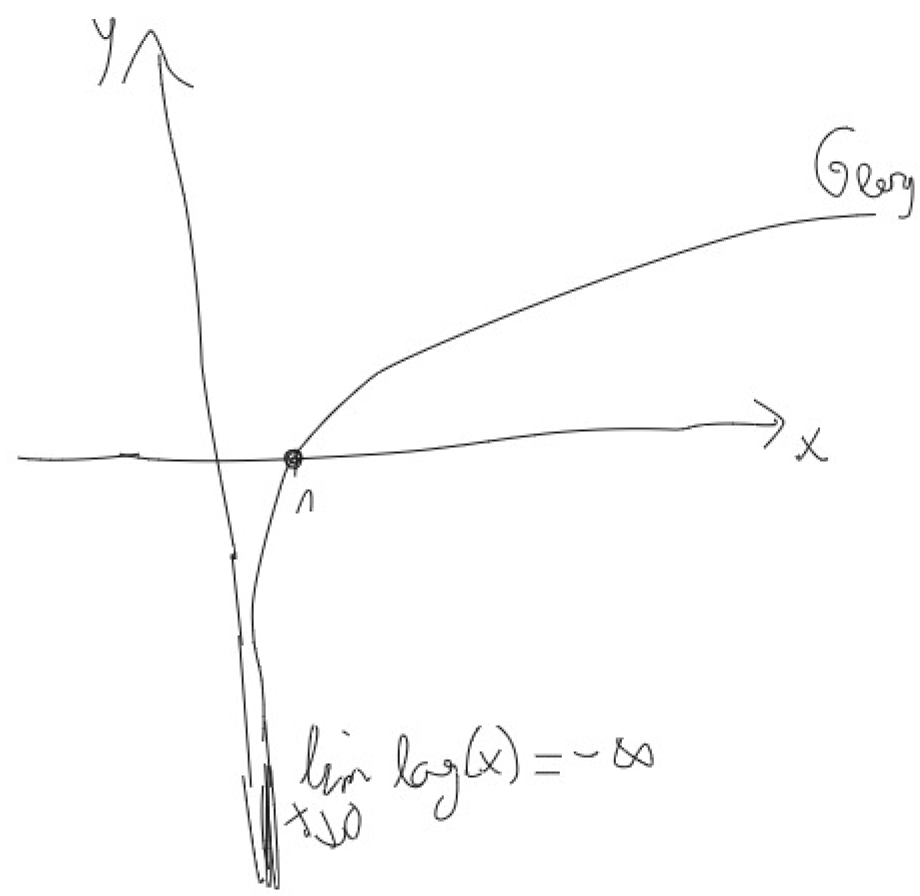
We are looking for $p_1, \dots, p_n \in (0,1)$ such that $H(X)$ is minimal.

$$\{(p_1, \dots, p_n) \in [0,1]^n \mid H(X) = 0, \sum_{i=1}^{n} p_i = 1\}$$
$$= \{(1,0,0,\dots,0), (0,1,0,\dots,0), \dots, (0,0,\dots,1)\}$$

Example: $\mathbb{P}(X = x_1) = p_1 = 1$ $\quad \mathbb{P}(X = x_i) = p_i = 0 \quad \forall i \in \{2, \dots, n\}$

$$H(X) = -p_1 \cdot \log(p_1) = -1 \cdot \underbrace{\log(1)}_{=0} = \underline{\underline{0}}$$

$$H(X) = -\underset{1}{p_1} \cdot \log(\underset{1}{p_1}) - \underbrace{\sum_{i=2}^{n} \underset{0}{p_i} \cdot \underbrace{\log(p_i)}_{=\infty}}_{=0} = \underline{\underline{0}}$$



$$\lim_{x \downarrow 0} \log(x) = -\infty$$

$$\lim_{x \downarrow 0} x \cdot \log(x) = 0$$

$$10^{-10} \cdot \log(10^{-10}) \approx$$

L'Hopital

$$\boxed{(\log x)' = \frac{1}{x}}$$

Swiss mathematician Johann Bernoulli.

L'Hôpital's rule states that for functions f and g which are differentiable on an open interval I except possibly at a point c contained in I, if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$, and $g'(x) \neq 0$ for all x in I with $x \neq c$, and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

The differentiation of the numerator and denominator often simplifies the quotient or converts it to a limit that can be evaluated directly.

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$$\lim_{x \downarrow 0} x \cdot \log(x) =$$

$$= \lim_{x \downarrow 0} \frac{\log(x)}{\frac{1}{x}} \overset{!}{=} \lim_{x \downarrow 0} \frac{(\log(x))'}{\left(\frac{1}{x}\right)'} = \lim_{x \downarrow 0} \frac{\cancel{\frac{1}{x}}}{(-1) \cdot \cancel{\frac{1}{x}} \cdot \frac{1}{x}} = \lim_{x \downarrow 0} (-1) \cdot x = \underline{\underline{0}}$$

$$\left(\frac{1}{x}\right)' = \left(x^{\boxed{-1}}\right)' = (-1) \cdot x^{-2} = (-1) \frac{1}{x^2}$$

$$\frac{1}{2} = 2^{-1}$$

$$\log\left(\frac{1}{2}\right) = -1$$

$\log\left(\frac{1}{2}\right) = -1$

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a)

$$H(Y) = H(2\cdot X) \overset{!}{=} H(X) \qquad 2\cdot H(X) \quad H(X)$$

<u>Proof</u> to prove

$$\mathcal{Y} = \{y \mid \mathbb{P}(Y=y) > 0\} = \{2\cdot x \mid \mathbb{P}(X=x) > 0\} = 2\cdot\mathcal{X}$$

$$H(Y) = \sum_{y\in\mathcal{Y}} \mathbb{P}(Y=y)\cdot\log(\mathbb{P}(Y=y))$$

$$= \sum_{x\in\mathcal{X}} \mathbb{P}(\cancel{2}\cdot X = \cancel{2}\cdot x)\cdot\log(\mathbb{P}(\cancel{2}\cdot X = \cancel{2}\cdot x))$$

$$= H(X)$$

$5x = 10 \quad |:5$

$x = 2$

<u>Example:</u> $\mathbb{P}(X=1) = \frac{1}{2} \quad \mathbb{P}(X=-1) = \frac{1}{2}$

$H(X) > 0$

$$H(X^2) = H(1) = 0$$

$H(X^2) \neq H(X)$

$H(2\cdot X) = H(X)$

Which property must f have such that

$$H(f(X)) = H(X) \; ?$$

Verben	
to draw a blank [fig.]	auf dem Schlauch stehen [fig.]
to be at stake	auf dem Spiel stehen